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## Degree of Non-Markovianity of Quantum Evolution

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We propose a new characterization of non-Markovian quantum evolution based on the concept of non-Markovianity degree. It provides an analog of a Schmidt number in the entanglement theory and reveals the formal analogy between quantum evolution and the entanglement theory: Markovian evolution corresponds to a separable state and the non-Markovian one is further characterized by its degree. It enables one to introduce a non-Markovianity witness—an analog of an entanglement witness, and a family of measures—an analog of Schmidt coefficients, and finally to characterize maximally non-Markovian evolution being an analog of the maximally entangled state. Our approach allows us to classify the non-Markovianity measures introduced so far in a unified rigorous mathematical framework.

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**Introduction.**—Open quantum systems and their dynamical features are attracting increasing attention nowadays. They are of paramount importance in the study of the interaction between a quantum system and its environment, causing dissipation, decay, and decoherence [1–3]. On the other hand, the robustness of quantum coherence and entanglement against the detrimental effects of the environment is one of the major focuses in quantum-enhanced applications, as both entanglement and quantum coherence are basic resources in modern quantum technologies, such as quantum communication, cryptography, and computation [4]. Recently, much effort was devoted to the description, analysis, and classification of non-Markovian quantum evolution (see, e.g., [5–19] and the collection of papers in [20]). In particular, various concepts of non-Markovianity were introduced and several so-called non-Markovianity measures were proposed. The main approaches to the problem of (non)Markovian evolution are based on divisibility [9–12], distinguishability of states [13], quantum entanglement [10], quantum Fisher information flow [14], fidelity [15], mutual information [16,17], channel capacity [18], and geometry of the set of accessible states [19].

In this Letter we accept the definition based on divisibility [9,10]: the quantum evolution is Markovian if the corresponding dynamical map  $\Lambda_t$  is CP divisible (where CP stands for complete positivity), that is,

$$\Lambda_t = V_{t,s} \Lambda_s, \quad (1)$$

and  $V_{t,s}$  provides a family of legitimate (completely positive and trace-preserving) propagators for all  $t \geq s \geq 0$ . The essential property of  $V_{t,s}$  is the following composition law  $V_{t,s} V_{s,u} = V_{t,u}$ , for all  $t \geq s \geq u$ . It provides a natural generalization of a semigroup law

$e^{tL} e^{sL} = e^{(t+s)L}$ . Interestingly, the very property of CP divisibility is fully characterized in terms of the time-local generator  $L_t$ : if  $\Lambda_t$  satisfies the time-local master equation  $\dot{\Lambda}_t = L_t \Lambda_t$ , then  $\Lambda_t$  is CP divisible if and only if  $L_t$  has the standard Lindblad form for all  $t \geq 0$ , i.e.,

$$L_t \rho = -i[H(t), \rho] + \sum_{\alpha} \left( V_{\alpha}(t) \rho V_{\alpha}^{\dagger}(t) - \frac{1}{2} \{ V_{\alpha}^{\dagger}(t) V_{\alpha}(t), \rho \} \right),$$

with time-dependent Lindblad (noise) operators  $V_{\alpha}(t)$  and time-dependent effective system Hamiltonian  $H(t)$  [3,21,22]. A very appealing concept of Markovianity was proposed by Breuer, Lane, and Piilo (BLP) [13]:  $\Lambda_t$  is Markovian if

$$\sigma(\rho_1, \rho_2; t) = \frac{d}{dt} \|\Lambda_t(\rho_1 - \rho_2)\|_1 \leq 0, \quad (2)$$

for all pairs of initial states  $\rho_1$  and  $\rho_2$ . BLP call  $\sigma(\rho_1, \rho_2; t)$  an information flow and interpret  $\sigma(\rho_1, \rho_2; t) > 0$  as a backflow of information from the environment to the system which clearly indicates the non-Markovian character of the evolution. As usual  $\|X\|_1$  denotes the trace norm of  $X$ , i.e.,  $\|X\|_1 = \text{Tr} \sqrt{XX^{\dagger}}$ . It turns out that CP divisibility implies (2) but the converse needs not be true [23–25].

In this Letter we propose a more refined approach to non-Markovian evolution. We reveal the formal analogy with the entanglement theory: Markovian evolution corresponds to a separable state and non-Markovian evolution is characterized by a positive integer—the non-Markovianity degree—corresponding to the Schmidt number of an entangled state. The notion of non-Markovianity degree enables one to introduce a family of measures and finally to

characterize maximally non-Markovian evolution being an analog of the maximally entangled state.

*Schmidt number and  $k$ -positive maps.*—Let us recall that a state of a composite quantum system may be uniquely characterized by its Schmidt number [26,27]: for any normalized vector  $\psi \in \mathcal{H} \otimes \mathcal{H}$  let  $\text{SR}(\psi)$  denote the Schmidt rank of  $\psi$ , i.e., a number of nonvanishing Schmidt coefficients in the decomposition  $\psi = \sum_k s_k e_k \otimes f_k$ , with  $s_k > 0$  and  $\sum_k s_k^2 = 1$ . Now, for any density operator  $\rho$  one defines its Schmidt number by

$$\text{SN}(\rho) = \min_{\rho_k, \psi_k} \{ \max_k \text{SR}(\psi_k) \}, \quad (3)$$

where the minimum is performed over all decompositions  $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$  with  $p_k > 0$  and  $\sum_k p_k = 1$ . Let  $S_k = \{\rho | \text{SN}(\rho) \leq k\}$ . One has  $S_1 \subset S_2 \subset \dots \subset S_n$ , where  $S_1$  denotes a set of separable states and  $S_n$  denotes a set of all states in  $\mathcal{H} \otimes \mathcal{H}$ . Note that a maximally entangled state  $\psi$  satisfies  $\lambda_1 = \dots = \lambda_n$  and the corresponding projector  $|\psi\rangle\langle\psi|$  defines an element of  $S_n$ . The Schmidt number does not increase under local operation, i.e.,  $\text{SN}(\mathcal{E}_1 \otimes \mathcal{E}_2 \rho) \leq \text{SN}(\rho)$ , where  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are arbitrary quantum channels. Moreover, if  $\Phi$  is a  $k$ -positive map, i.e.,  $\mathbb{1}_k \otimes \Phi$  is positive, then for any  $\rho \in S_k$  one has  $[\mathbb{1}_k \otimes \Phi](\rho) \geq 0$  ( $\mathbb{1}_k$  denotes an identity map acting in  $M_k$ —the space of  $k \times k$  complex matrices). This simple property establishes a duality between  $k$ -positive maps and quantum bipartite states with the Schmidt number bounded by  $k$ .

*Non-Markovianity degree.*—The notion of  $k$ -positive maps enables one to provide a natural generalization of CP divisibility: we call a dynamical map  $\Lambda_t$   $k$  divisible if and only if  $V_{t,s}$  is  $k$  positive for all  $t \geq s \geq 0$ . Hence,  $n$ -divisible maps are CP divisible and 1 divisible are simply  $P$  divisible; i.e.,  $V_{t,s}$  is positive. Now, we introduce a degree of non-Markovianity which is an analog of a Schmidt number: a dynamical map  $\Lambda_t$  has a non-Markovianity degree  $\text{NMD}[\Lambda_t] = k$  if and only if  $\Lambda_t$  is  $(n-k)$  but not  $(n+1-k)$  divisible. It is clear that  $\Lambda_t$  is Markovian if and only if  $\text{NMD}[\Lambda_t] = 0$  and essentially non-Markovian if and only if  $\text{NMD}[\Lambda_t] = n$ . Denoting by  $\mathcal{N}_k = \{\Lambda_t | \text{NMD}[\Lambda_t] \leq k\}$ , one has a natural chain of inclusions

$$\mathcal{N}_0 \subset \mathcal{N}_1 \subset \dots \subset \mathcal{N}_{n-1} \subset \mathcal{N}_n, \quad (4)$$

where  $\mathcal{N}_0$  denotes Markovian maps and  $\mathcal{N}_n$  all dynamical maps. The characterization of  $k$ -divisible maps is provided by the following.

*Theorem 1.*—If  $\Lambda_t$  is  $k$  divisible, then

$$\frac{d}{dt} \|[\mathbb{1}_k \otimes \Lambda_t](X)\|_1 \leq 0, \quad (5)$$

for all operators  $X \in M_k \otimes \mathcal{B}(\mathcal{H})$ .

For the proof see Supplemental Material [28]. In particular, all  $k$ -divisible maps ( $k = 1, \dots, n$ ) satisfy

$$\frac{d}{dt} \|\Lambda_t(X)\|_1 \leq 0, \quad (6)$$

for all  $X \in \mathcal{B}(\mathcal{H})$ . Note that BLP condition (2) is a special case of (6) with  $X$  being traceless Hermitian operator. It is, therefore, clear that BLP condition is weaker than all conditions in the hierarchy (5) and it is satisfied for all  $k$ -divisible maps not necessarily CP divisible. According to our definition of Markovianity (Markovianity = CP divisibility)  $k$ -divisible maps which are not CP divisible are clearly non-Markovian. However, such non-Markovian evolutions always satisfy (6). We propose to call such dynamical maps *weakly* non-Markovian. A dynamical map which is even not  $P$  divisible will be called *essentially* non-Markovian. Hence,  $\Lambda_t$  is weakly non-Markovian if and only if  $\Lambda_t \in \mathcal{N}_{n-1} - \mathcal{N}_0$  and it is essentially non-Markovian if and only if  $\Lambda_t \in \mathcal{N}_n - \mathcal{N}_{n-1}$ . Using the notion of degree of non-Markovianity  $\Lambda_t$  is weakly non-Markovian if and only if  $0 < \text{NMD}[\Lambda_t] \leq n-1$  and it is essentially non-Markovian if and only if  $\text{NMD}[\Lambda_t] = n$ . Note that maps which violate the BLP condition are always essentially non-Markovian. Similarly, if  $\Lambda_t$  is at least 2 divisible, then the relative entropy satisfies the following monotonicity property [29]

$$\frac{d}{dt} S[\Lambda_t(\rho_1) | \Lambda_t(\rho_2)] \leq 0, \quad (7)$$

for any pair  $\rho_1$  and  $\rho_2$ . The violation of (7) means that  $\Lambda_t$  is at most  $P$  divisible or essentially non-Markovian. It should be stressed that there is crucial difference between CP divisibility and only  $k$  divisibility with  $k < n$ . CP divisibility guarantees that  $V_{t,s}$  are completely positive and, hence, they may be considered as physical propagators for  $s \leq t$ . This is no longer true for  $V_{t,s}$  which are not CP but only  $k$  positive. There was an active debate whether or not one can describe quantum evolution by maps which are more general than CP maps [30]. Usually, the departure from complete positivity is attributed to the presence of initial system-environment correlations [30]. Remarkably, in our approach the lack of complete positivity of  $V_{t,s}$  corresponds to memory effects caused by the nontrivial system-environment interaction. We stress that the dynamical map  $\Lambda_t$  is perfectly CP; only the intermediate propagators  $V_{t,s}$  are not. Note, however, that if  $\Lambda_t$  is  $k$  divisible then  $V_{t,s}$  map a state in time  $s$  into a state in time  $t$ . One loses this property only if  $\Lambda_t$  is essentially non-Markovian.

*Non-Markovianity witness.*—Actually, if  $\Lambda_t$  is invertible, then it is  $k$  divisible if and only if (5) holds. Clearly, a generic map is invertible (all its eigenvalues are different from zero) and hence this result is true for a generic dynamical map (a notable exception is the Jaynes-Cummings model on resonance [1,31]). Hence, if (5) is violated for some  $t > 0$ , then  $\Lambda_t$  is not  $k$  divisible or, equivalently,  $\text{NMD}[\Lambda_t] > n-k$ . It is, therefore, natural to call such  $X$  a non-Markovianity witness in analogy to

the well-known concept of an entanglement witness. Recall, that a Hermitian operator  $W$  living in  $\mathcal{H} \otimes \mathcal{H}$  is an entanglement witness [26] if and only if (i)  $\langle \Psi | W | \Psi \rangle \geq 0$  for all product vectors  $\Psi = \psi \otimes \phi$ , and (ii)  $W$  is not a positive operator; i.e., it possesses at least one negative eigenvalue. Similarly,  $W$  is a  $k$ -Schmidt witness [32] if  $\langle \Psi | W | \Psi \rangle \geq 0$  for all vectors  $\Psi = \psi_1 \otimes \phi_1 + \dots + \psi_k \otimes \phi_k$ , that is, if  $\text{Tr}(\rho W) < 0$ , then  $\rho$  is entangled and moreover  $\text{SN}(\rho) > k$ . Note, that if  $X \geq 0$ , then (5) is always satisfied due to the fact that  $\|[\mathbb{1}_k \otimes \Lambda_t](X)\|_1 = \|X\|_1$ . Hence, similarly as  $W$ , a non-Markovianity witness  $X$  has to possess a negative eigenvalue.

*Non-Markovianity measures.*—The above construction allows us to define a series of natural measures measuring departure from  $k$  divisibility,

$$\mathcal{M}_k[\Lambda_t] = \sup_X \frac{N_k^+[X]}{|N_k^-[X]|}, \quad (8)$$

where

$$N_k^+[X] = \int_{\lambda_k(X;t) > 0} \lambda_k(X;t) dt,$$

and, similarly for  $N_k^-[X]$  (where now one integrates over time intervals such that  $\lambda_k(X;t) < 0$ ), and

$$\lambda_k(X;t) = \frac{d}{dt} \|[\mathbb{1}_k \otimes \Lambda_t](X)\|_1. \quad (9)$$

The supremum is taken over all Hermitian  $X \in M_k \otimes \mathcal{B}(\mathcal{H})$ . Note that

$$\begin{aligned} & \int_0^\infty \frac{d}{dt} \|[\mathbb{1}_k \otimes \Lambda_t](X)\|_1 dt \\ &= \|[\mathbb{1}_k \otimes \Lambda_\infty](X)\|_1 - \|X\|_1 \leq 0, \end{aligned}$$

and hence  $|N_-[\Lambda_t]| \geq N_+[\Lambda_t]$ , which proves that  $\mathcal{M}_k[\Lambda_t] \in [0, 1]$ . Clearly, if  $l > k$ , then  $\mathcal{M}_l[\Lambda_t] \geq \mathcal{M}_k[\Lambda_t]$  and, hence,

$$0 \leq \mathcal{M}_1[\Lambda_t] \leq \dots \leq \mathcal{M}_n[\Lambda_t] \leq 1,$$

which provides an analog of a similar relation among the Schmidt coefficients  $s_1 \geq \dots \geq s_n$ . Now, following the analogy with an entanglement theory, we may call  $\Lambda_t$  maximally non-Markovian if and only if  $\mathcal{M}_1[\Lambda_t] = 1$ , which immediately implies

$$\mathcal{M}_1[\Lambda_t] = \dots = \mathcal{M}_n[\Lambda_t] = 1, \quad (10)$$

in a perfect analogy with maximally entangled state corresponding to  $s_1 = \dots = s_n$ .

*Examples.*—Let us illustrate the above introduced notions by a few simple examples.

**Example 1:** Consider pure decoherence of a qubit system described by the following local generator

$$L_t(\rho) = \frac{1}{2}\gamma(t)(\sigma_z \rho \sigma_z - \rho), \quad (11)$$

The corresponding evolution of the density matrix reads

$$\rho_t = \begin{pmatrix} \rho_{11} & \rho_{12}e^{-\Gamma(t)} \\ \rho_{12}e^{-\Gamma(t)} & \rho_{22} \end{pmatrix}, \quad (12)$$

where  $\Gamma(t) = \int_0^t \gamma(\tau) d\tau$ . The evolution is completely positive if and only if  $\Gamma(t) \geq 0$  and it is  $k$  divisible ( $k = 1, 2$ ) if and only if  $\gamma(t) \geq 0$ . Taking  $X = \sigma_x$  one finds  $\|\Lambda_t(X)\|_1 = 2e^{-\Gamma(t)}$ . Observe that

$$|N_-[\Lambda_t]| = N_+[\Lambda_t] + e^{-\Gamma(\infty)} - 1, \quad (13)$$

and hence if  $\Gamma(\infty) = 0$  the evolution is maximally non-Markovian. Note, that  $\Gamma(\infty) = 0$  implies that  $\rho_t \rightarrow \rho$ , that is, asymptotically one always recovers an initial state—perfect recoherence. Actually, this example may be immediately generalized as follows: let  $L$  be a Lindblad generator and consider a time-dependent generator defined by  $L_t = \gamma(t)L$ . Now,  $L_t$  gives rise to a legitimate quantum dynamical map if and only if  $\Gamma(t) \geq 0$  and it is  $k$  divisible ( $k = 1, 2, \dots, n$ ) if and only if  $\gamma(t) \geq 0$ . The corresponding dynamics is maximally non-Markovian if  $\Gamma(\infty) = 0$ .

**Example 2:** Consider the qubit dynamics governed by the time-dependent generator

$$L_t(\rho) = \frac{1}{2} \sum_{k=1}^3 \gamma_k(t)(\sigma_k \rho \sigma_k - \rho). \quad (14)$$

It is clear that (14) provides a simple generalization of (11) by introducing two additional decoherence channels. The corresponding dynamical map reads

$$\Lambda_t(\rho) = \sum_{\alpha=0}^3 p_\alpha(t) \sigma_\alpha \rho \sigma_\alpha, \quad (15)$$

where  $\sigma_0 = \mathbb{I}$ , and the probability distribution  $p_\alpha(t)$  may be easily calculated in terms of  $\gamma_k(t)$  (see Ref. [33]). Interestingly, in this example there is an essential difference between CP divisibility (= Markovianity) and only  $P$  divisibility: CP divisibility is equivalent to

$$\gamma_1(t) \geq 0, \quad \gamma_2(t) \geq 0, \quad \gamma_3(t) \geq 0, \quad (16)$$

whereas  $P$  divisibility is equivalent to much weaker conditions [33]

$$\begin{aligned} \gamma_1(t) + \gamma_2(t) &\geq 0, \\ \gamma_1(t) + \gamma_3(t) &\geq 0, \\ \gamma_2(t) + \gamma_3(t) &\geq 0, \end{aligned} \quad (17)$$

for all  $t \geq 0$ . Actually, the BLP condition reproduces (17). Now, violation of at least one inequality from (17) implies essential non-Markovianity. Suppose for example that  $\gamma_2(t) + \gamma_3(t) \not\geq 0$ . Assuming that  $\Gamma_2(\infty) = \Gamma_3(\infty) = 0$

one finds that  $\mathcal{M}_1[\Lambda_t] = 1$ , that is,  $\Lambda_t$  is maximally non-Markovian. Interestingly, if there are at most two decoherence channels, then there is no difference between CP and  $P$  divisibility. Note, that random unitary dynamics (15) is unital; i.e.,  $\Lambda_t(\mathbb{I}) = \mathbb{I}$  and, hence, during the evolution the entropy never decreases  $S[\Lambda_t(\rho)] \geq S(\rho)$  for any initial qubit state  $\rho$ . One easily shows that  $P$  divisibility is equivalent to

$$\frac{d}{dt} S[\Lambda_t(\rho)] \geq 0, \quad (18)$$

for any qubit state  $\rho$ . Hence, for any weakly non-Markovian random unitary dynamics the von Neumann entropy monotonically increases. Violation of (18) proves that  $\Lambda_t$  is essentially non-Markovian.

Example 3: Consider a qubit dynamics governed by the following local generator

$$L_t = \gamma_+(t)L_+ + \gamma_-(t)L_-, \quad (19)$$

where  $L_+(\rho) = \frac{1}{2}([\sigma_+, \rho\sigma_-] + [\sigma_+\rho, \sigma_-])$  and  $L_-(\rho) = \frac{1}{2}([\sigma_-, \rho\sigma_+] + [\sigma_-\rho, \sigma_+])$ , with  $\sigma_+ = |2\rangle\langle 1|$  and  $\sigma_- = |1\rangle\langle 2|$ .  $L_+$  generates pumping from the ground state  $|1\rangle$  to an excited state  $|2\rangle$  and  $L_-$  generates a decay from  $|2\rangle$  to  $|1\rangle$ . One shows that  $L_t$  generates legitimate dynamical map if and only if

$$0 \leq \int_0^t \gamma_{\pm}(s) e^{\Gamma(s)} ds \leq e^{\Gamma(t)} - 1, \quad (20)$$

where  $\Gamma(t) = \int_0^t [\gamma_-(\tau) + \gamma_+(\tau)] d\tau$ . In particular, it follows from (20) that  $\Gamma(t) \geq 0$ . Now,  $\Lambda_t$  is CP divisible if and only if

$$\gamma_-(t) \geq 0, \quad \gamma_+(t) \geq 0, \quad (21)$$

and it is  $P$  divisible if and only if

$$\gamma_-(t) + \gamma_+(t) \geq 0. \quad (22)$$

Note, that (21) implies (20). However, it is not true for (22): i.e.,  $P$  divisibility requires both (20)—it guarantees that  $\Lambda_t$  is completely positive—and (22).

*Bloch equations and  $P$  divisibility.*—The above examples illustrating qubit dynamics may be easily rewritten in terms of the Bloch vector  $x_k(t) = \text{Tr}[\sigma_k \Lambda_t(\rho)]$ . Example 2 gives rise to

$$\frac{d}{dt} x_k(t) = -\frac{1}{T_k(t)} x_k(t), \quad k = 1, 2, 3, \quad (23)$$

where  $T_1(t) = [\gamma_2(t) + \gamma_3(t)]^{-1}$ , and similarly for  $T_2(t)$  and  $T_3(t)$ . Quantities  $T_k(t)$  correspond to local relaxation times. It is therefore clear that  $P$  divisibility is equivalent to  $T_k(t) \geq 0$  for  $k = 1, 2, 3$ . This proves the essential difference between CP divisibility and  $P$  divisibility. CP

divisibility requires that all local decoherence rates satisfy  $\gamma_k(t) \geq 0$ , whereas  $P$  divisibility requires only  $T_k(t) \geq 0$ . Hence, one may have temporarily negative decoherence rates but always positive relaxation times. From a physical point of view this shows that the two main non-Markovianity measures used in the literature describe very different ways in which memory effects manifest themselves. Violation of CP divisibility [10] reflects the presence of reverse quantum jumps [6], restoring previously lost coherence and occurring when one of the decay rates becomes negative. The BLP non-Markovianity [13], which in this case corresponds to the violation of  $P$  divisibility, instead occurs when at least one of the relaxation times becomes temporarily negative; i.e., instead of relaxation one of the components  $x_k(t)$  temporarily grows. This in turn stems from a temporary and partial increase of information on the open system, as measured by trace distance. Note, that CP divisibility is equivalent to  $P$  divisibility plus three extra conditions

$$\frac{1}{T_1} + \frac{1}{T_2} \geq \frac{1}{T_3}, \quad \frac{1}{T_1} + \frac{1}{T_3} \geq \frac{1}{T_2}, \quad \frac{1}{T_2} + \frac{1}{T_3} \geq \frac{1}{T_1}.$$

Finally, let us observe that the initial volume of the Bloch ball shrinks during the evolution according to

$$V(t) = e^{-[\Gamma_1(t) + \Gamma_2(t) + \Gamma_3(t)]} V(0),$$

where  $V(t)$  denotes a volume of the set of accessible states at time  $t$ . Authors of [19] characterized non-Markovian evolution as a departure from  $(d/dt)V(t) \leq 0$ . One has  $(d/dt)V(t) = -[\gamma_1(t) + \gamma_2(t) + \gamma_3(t)]V(t)$  and, hence,  $(d/dt)V(t) \leq 0$  if and only if

$$\gamma_1(t) + \gamma_2(t) + \gamma_3(t) \geq 0. \quad (24)$$

This condition is much weaker than (17). To violate (24) the evolution has to be essentially non-Markovian (i.e.,  $\Lambda_t$  cannot be even  $P$  divisible). Actually, the geometric condition of [19] is always weaker than  $P$  divisibility (and hence  $k$  divisibility). Any  $k$ -divisible dynamics necessarily satisfies  $(d/dt)V(t) \leq 0$ .

A similar conclusion may be drawn from Example 3: the corresponding Bloch equations read

$$\begin{aligned} \frac{d}{dt} x_k(t) &= -\frac{1}{T_{\perp}(t)} x_k(t), \quad k = 1, 2, \\ \frac{d}{dt} x_3(t) &= -\frac{1}{T_{\parallel}(t)} x_3(t) + \Delta(t), \end{aligned} \quad (25)$$

where  $\Delta(t) = [\gamma_+(t) - \gamma_-(t)]$ , and  $T_{\perp}(t) = 2/[\gamma_-(t) + \gamma_+(t)]$  and  $T_{\parallel}(t) = T_{\perp}(t)/2$  are transverse and longitudinal local relaxation times, respectively. Again,  $P$  divisibility is equivalent to  $T_{\perp}, T_{\parallel}(t) \geq 0$ , provided that the Bloch vector stays within a Bloch ball.



**Conclusions.**—In this Letter we provided further characterization of non-Markovian evolution in terms of the non-Markovianity degree. This simple concept, being an analog of the Schmidt number in the entanglement theory, enables one to compare quantum evolutions. We say that  $\Lambda_t^{(1)}$  is more non-Markovian than  $\Lambda_t^{(2)}$  if  $\text{NMD}[\Lambda_t^{(1)}] > \text{NMD}[\Lambda_t^{(2)}]$ . Similarities and differences between the existing non-Markovianity measures in specific open system models have been discussed in several papers [18,19,23–25,31,34,35]. However, their general connection was still an open problem. Here we have solved this problem in full generality by defining a hierarchy of non-Markovianity measures. They interpolate between the well-known RHP [10] and BLP [13] measures and, hence, they provide refinement of non-Markovianity measures usually used in the literature. This way our approach allows us to classify the non-Markovianity measures introduced so far in a unified framework. Finally, we define a notion of maximally non-Markovian evolution which is an analog of a maximally entangled state. Maximally, non-Markovian evolution may be of crucial importance if non-Markovianity can be shown to be a resource for quantum technologies, as recent results suggest [18]. Finally, if the evolution is only  $k$  divisible with  $k < n$  one may ask about additional properties of the dynamical map. In particular an interesting issue might be the optimality of the family of “propagators” [36]  $V_{t,s}$ .

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